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MATHEMATICAL MODELING OF THE PROCESS OF HEAT TRANSFER IN A SHIELDED HALF-SPACE WITH A THERMOACTIVE SPACER UNDER EXTERNAL THERMAL ACTION

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A refined "lumped-capacity" model has been proposed for description of the process of formation of a temperature field in the isotropic half-space–thermoactive spacer–thermal protective coating system exposed to the external heat flux; the determining parameter of this model has been identified and the possibility of using it in practice has been substantiated.

Investigations of the processes of formation of temperature fields in conjugate solid bodies occupy a special place in applications of mathematical heat-conduction theory [1–4]. Their results are important in solving the problem (of practical importance) of development of the efficient methods of thermal protection of structures [5–8]. The application of thermoelectric phenomena (Peltier and Thompson effects [9, 10]) to control the temperature state of a structure or to its thermostatting is promising in this field. One basic element of a thermoelectric thermal-protection system is a thermoactive spacer (gasket) [11], in which we have the release or absorption of heat with a prescribed specific power as a result of external controlled actions.

In the simplest case the structure is simulated by an isotropic half-space, and a parametric analysis of the temperature field in the half-space with a thermoelectric thermal-protection system is made with a mathematical model containing specific conjugation conditions [11]. In the general case these conditions are nonstationary and ensure the equality of temperatures on the contact surface and the discontinuity of heat fluxes [11]. The difficulties to be surmounted in parametric analysis of the temperature field in the system under study are well known, as are assumptions of different kinds [11–15] whose realization leads to simplified analogs of the mathematical models used.

In [11], we have proposed and substantiated a mathematical model of the process of formation of a temperature field in the isotropic half-space–thermoactive thermally thin spacer–thermal protective coating system exposed to the external heat flux. The development of a simplified analog of this model was the prime objective of the investigations carried out. As the optimality criterion of the simplified analog of the initial model, we used the minimax criterion for the modulus of the difference of integral values of the heat loss [12, 15] in the thermal protective coating which had been determined with the initial and simplified models. This enabled us to use the one-dimensional model of the process studied as the object of investigations.

Initial Model and Its Simplified Analog. A one-dimensional mathematical model of the process of formation of a temperature field in the isotropic half-space–thermoactive thermally thin spacer with a constant specific heat-absorption power Q_s–thermal protective coating system exposed to the nonstationary heat flux Q_0 (Fig. 1) can be represented in the form [13]

$$\frac{\partial \theta (y, \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{\partial^2 \theta (y, \mathrm{Fo})}{\partial y^2}, \quad \mathrm{Fo} > 0, \quad y > \varepsilon ; \tag{1}$$

$$\varepsilon_* \frac{d \langle \theta (\mathrm{Fo}) \rangle}{d\mathrm{Fo}} = \left. \frac{\partial \theta (y, \mathrm{Fo})}{\partial y} \right|_{y=\varepsilon+0} - \Lambda \left. \frac{\partial \theta (y, \mathrm{Fo})}{\partial y} \right|_{y=0-0} - Q_s , \quad \mathrm{Fo} > 0, \quad 0 < y < \varepsilon ; \tag{2}$$

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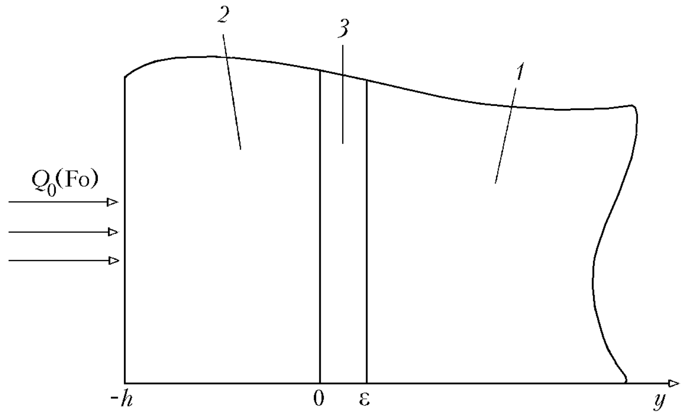


Fig. 1. Diagram of the analyzed system: 1) isotropic half-space; 2) thermal protective coating; 3) thermoactive spacer.

$$\frac{\partial \theta (y, \mathrm{Fo})}{\partial \mathrm{Fo}} = \chi^2 \frac{\partial^2 \theta (y, \mathrm{Fo})}{\partial y^2}, \quad \mathrm{Fo} > 0, \quad -h < y < 0 ; \tag{3}$$

$$\theta (y, \mathrm{Fo}) \big|_{\mathrm{Fo}=0} = \langle \theta (\mathrm{Fo}) \rangle \big|_{\mathrm{Fo}=0} = 0 ; \tag{4}$$

$$\left. \frac{\partial \theta (y, \mathrm{Fo})}{\partial y} \right|_{y=-h} = -\frac{Q_0}{\Lambda} ; \tag{5}$$

$$\theta (y, \mathrm{Fo}) \big|_{y=\varepsilon+0} = \langle \theta (\mathrm{Fo}) \rangle = \theta (y, \mathrm{Fo}) \big|_{y=0-0} ; \tag{6}$$

$$\theta (y, \mathrm{Fo}) \big|_{\mathrm{Fo}>0} \in L^2 [\varepsilon, +\infty) , \tag{7}$$

where the last condition means that the function $\theta(y, \mathrm{Fo})$ is integrable with square in the spatial variable $y \in [\varepsilon, +\infty]$ for each fixed $\mathrm{Fo} > 0$.

In the mathematical model (1)–(7), we have

$$\theta = \frac{T - T_0}{T_0}, \quad y = \frac{z}{z_*}, \quad z_* = l_2 \sqrt{a_1/a_2}, \quad \mathrm{Fo} = \frac{a_1 t}{z_*^2}, \quad \varepsilon = \frac{l_3}{z_*}, \quad \varepsilon_* = \varepsilon \frac{\lambda_3 a_1}{\lambda_1 a_3}, \quad \Lambda = \frac{\lambda_2}{\lambda_1},$$

$$Q_s = \frac{\overline{W} z_*}{\lambda_1 T_0}, \quad \chi^2 = \frac{a_2}{a_1}, \quad h = \frac{l_2}{z_*}, \quad Q_0 = \frac{q_0 z_*}{\lambda_1 T_0}, \quad \langle \theta (\mathrm{Fo}) \rangle = \frac{1}{\varepsilon} \int_0^{\varepsilon} \theta (y, \mathrm{Fo}) \, dy .$$

For the purpose in hand we introduce into consideration the temperature mean-integral over the thermal protective coating

$$\bar{\theta} (\mathrm{Fo}) = \frac{1}{h} \int_{-h}^{0} \theta (y, \mathrm{Fo}) \, dy \tag{8}$$

and, according to (3) and (8), arrive at the equation

$$\frac{d\bar{\theta}(\text{Fo})}{d\text{Fo}} = \frac{\chi^2}{h}\left\{\frac{\partial\theta(y,\text{Fo})}{\partial y}\bigg|_{y=0-0} - \frac{\partial\theta(y,\text{Fo})}{\partial y}\bigg|_{y=-h+0}\right\},$$

which, with account for (2) and (5), can be transformed to the following form:

$$\text{K}_\varepsilon \frac{d\bar{\theta}(\text{Fo})}{d\text{Fo}} = \frac{\partial\theta(y,\text{Fo})}{\partial y}\bigg|_{y=\varepsilon+0} - \varepsilon_* \frac{d\langle\theta(\text{Fo})\rangle}{d\text{Fo}} + [Q_0 - Q_s], \tag{9}$$

where the criterion $\text{K}_\varepsilon = \lambda_2\sqrt{a_1}/(\lambda_1\sqrt{a_2})$ characterizes the thermal activity of the thermal protective coating in relation to the half-space and takes on values on the half-open interval (0; 1] [2]. If we assume that heat-exchange in the half-space–thermal-protective coating system is realized by the Newton law with an unknown parameter μ [12, 14, 15]:

$$\frac{\partial\theta(y,\text{Fo})}{\partial y}\bigg|_{y=\varepsilon+0} = \mu\left\{\theta(y,\text{Fo})\big|_{y=\varepsilon+0} - \bar{\theta}(\text{Fo})\right\},$$

then, taking account of (1), (4), (6), (9), and (7), we arrive at a simplified analog of the initial model (1)–(7):

$$\frac{\partial\theta(y,\text{Fo})}{\partial\text{Fo}} = \frac{\partial^2\theta(y,\text{Fo})}{\partial y^2}, \quad \text{Fo} > 0, \quad y > \varepsilon;$$

$$\text{K}_\varepsilon \frac{d\bar{\theta}(\text{Fo})}{d\text{Fo}} = \frac{\partial\theta(y,\text{Fo})}{\partial y}\bigg|_{y=\varepsilon+0} - \varepsilon_* \frac{\partial\theta(y,\text{Fo})}{\partial\text{Fo}}\bigg|_{y=\varepsilon+0} + [Q_0 - Q_s], \quad \text{Fo} > 0;$$

$$\bar{\theta}(\text{Fo})\big|_{\text{Fo}=0} = \theta(y,\text{Fo})\big|_{\text{Fo}=0} = 0;$$

$$\frac{\partial\theta(y,\text{Fo})}{\partial y}\bigg|_{y=\varepsilon+0} = \mu\left\{\theta(y,\text{Fo})\big|_{y=\varepsilon+0} - \bar{\theta}(\text{Fo})\right\};$$

$$\theta(y,\text{Fo})\big|_{\text{Fo}>0} \in L^2[\varepsilon, +\infty). \tag{10}$$

Following the established terminology [12, 14, 15], the initial mathematical model (1)–(7) of the process under study will be called the "accurate" model, whereas its simplified analog (10) will be referred to as the refined "lumped-capacity" model. Also, we note that the parameter μ is to be identified, and an equivalent representation of the mathematical model (10) is possible:

$$\frac{\partial\theta(y,\text{Fo})}{\partial\text{Fo}} = \frac{\partial^2\theta(y,\text{Fo})}{\partial y^2}, \quad \text{Fo} > 0, \quad y > \varepsilon;$$

$$(\text{K}_\varepsilon + \varepsilon_*)\frac{\partial\theta(y,\text{Fo})}{\partial\text{Fo}} = \frac{\text{K}_\varepsilon}{\mu}\frac{\partial^2\theta(y,\text{Fo})}{\partial\text{Fo}\,\partial y} + \frac{\partial\theta(y,\text{Fo})}{\partial y} + [Q_0 - Q_s], \quad \text{Fo} > 0, \quad y = \varepsilon + 0;$$

$$\theta(y,\text{Fo})\big|_{\text{Fo}=0} = 0; \quad \theta(y,\text{Fo})\big|_{\text{Fo}>0} \in L^2[\varepsilon, +\infty). \tag{11}$$

Identification of the Parameter μ. For the sake of convenience we introduce the subscript $j \in \{1, 2\}$ and will assume that $j = 1$ corresponds to the "accurate" model (1)–(7), whereas $j = 2$ corresponds to the refined "lumped-capacity" model (11). Thus, in what follows, the function $\theta_j(y, \text{Fo})$ determines the temperature field under study with the initial mathematical model (1)–(7) for $j = 1$ and its simplified analog (11) for $j = 2$.

To solve the problem of identification of the parameter μ contained in the mathematical model (11) we use the existing approach [12, 15]. For its realization we introduce into consideration the following functionals:

$$Q_{\mathrm{p}}(\mathrm{Fo}) = -\int_0^{\mathrm{Fo}} \left.\frac{\partial\theta(y,\tau)}{\partial y}\right|_{y=-h} d\tau\,, \tag{12}$$

the quantity of heat, put into the thermal protective coating over the period Fo > 0;

$$Q_{\mathrm{h}j}(\mathrm{Fo}) = -\int_0^{\mathrm{Fo}} \left.\frac{\partial\theta_j(y,\tau)}{\partial y}\right|_{y=\varepsilon+0} d\tau\,, \tag{13}$$

the quantity of heat received by the shielded half-space over the period Fo and determined with the jth model, where $j \in \{1, 2\}$;

$$W_j = \frac{Q_{\mathrm{h}j}}{Q_{\mathrm{p}}}\,,\quad j \in \{1, 2\} \tag{14}$$

are the integral values of the heat loss [12] in the thermal protective coating, determined with the initial model ($j = 1$) and its simplified analog ($j = 2$). Since $W_1 = W_1(\mathrm{Fo}, \gamma)$ and $W_2 = W_2(\mathrm{Fo}, \gamma, \mu)$, where $\gamma = [\varepsilon_*, \Lambda, \chi]^{\mathrm{t}} \in \Gamma \subset \mathbb{R}^3$ is the vector of determining parameters of the initial model (1)–(7) and its simplified analog (11), the problem of identification of the parameter μ can be represented in the form of a minimax optimization problem:

$$\max_{\substack{\mathrm{Fo}>0\\ \gamma\in\Gamma}} \left| W_1(\mathrm{Fo}, \gamma) - W_2(\mathrm{Fo}, \gamma, \mu) \right| \to \min_{\mu}\,. \tag{15}$$

To determine the integral values of the heat loss $\{W_j\}_{j=1}^2$ in the thermal protective coating of the considered system half-space–thermoactive thermally thin spacer–thermal protective coating, we represent the initial mathematical model (1)–(7) and its simplified analog (11) in the image space of the integral Laplace transform by a parameter $s \in \mathbb{C}$ [2]:

$$U_j(y, s) = L[\theta_j(y, \mathrm{Fo})] \equiv \int_0^{\infty} \exp(-s\mathrm{Fo})\,\theta_j(y, \mathrm{Fo})\, d\mathrm{Fo}\,,\quad j \in \{1, 2\}\,;$$

$$U(s) = L[\langle\theta(\mathrm{Fo})\rangle]\,. \tag{16}$$

In the image space, the "accurate" model ($j = 1$) can be represented as follows:

$$sU_1(y, s) = \frac{d^2U_1(y, s)}{dy^2}\,,\quad y > \varepsilon\,;$$

$$\varepsilon_* sU(s) = \left.\frac{dU_1(y, s)}{dy}\right|_{y=\varepsilon+0} - \Lambda \left.\frac{dU_1(y, s)}{dy}\right|_{y=0-0} - \frac{Q_{\mathrm{s}}}{s}\,;$$

$$\frac{s}{\chi^2} U_1(y, s) = \frac{d^2U_1(y, s)}{dy^2}\,,\quad -h < y < 0\,;\quad \left.\frac{dU_1(y, s)}{dy}\right|_{y=-h} = -\frac{Q_0}{s\Lambda}\,;$$

$$U_1(y,s)\big|_{y=\varepsilon+0} = U(s) = U_1(y,s)\big|_{y=0-0}\,; \quad U_1(y,s)\big|_{s\in\mathbb{C}} \in L^2[\varepsilon,+\infty)\,.$$

It unambiguously determines the integral Laplace transform (16) of the temperature field under study, in particular:

$$U_1(y,s)\big|_{y\geq\varepsilon} = \frac{\left\{2Q_0 \exp(-\sqrt{s}) - Q_s[1+\exp(-2\sqrt{s})]\right\} \exp[-(y-\varepsilon)\sqrt{s}]}{s\sqrt{s}\left\{1+\varepsilon_*\sqrt{s}+K_\varepsilon\right\}\left\{\dfrac{1+\varepsilon_*\sqrt{s}-K_\varepsilon}{1+\varepsilon_*\sqrt{s}+K_\varepsilon}\exp(-2\sqrt{s})+1\right\}}\,. \qquad (17)$$

In the image space, the refined "lumped-capacity" model ($j = 2$) represented in the form (11) has the following form:

$$sU_2(y,s) = \frac{d^2U_2(y,s)}{dy^2}\,, \quad y > \varepsilon\,;$$

$$(K_\varepsilon+\varepsilon_*)\, sU_2(y,s) = \left\{\frac{K_\varepsilon}{\mu}s+1\right\}\frac{dU_2(y,s)}{dy} + \frac{Q_0-Q_s}{s}\,, \quad y = \varepsilon+0\,;$$

$$U_2(y,s)\big|_{s\in\mathbb{C}} \in L^2[\varepsilon,+\infty)\,.$$

It also unambiguously determines the integral Laplace transform (16) of the temperature field studied but corresponding to the simplified analog of the initial model now (1)–(7). In particular,

$$U_2(y,s)\big|_{y\geq\varepsilon} = \frac{Q_0-Q_s}{s\sqrt{s}\left(\dfrac{K_\varepsilon}{\mu}s+(K_\varepsilon+\varepsilon_*)\sqrt{s}+1\right)}\exp[-(y-\varepsilon)\sqrt{s}]\,. \qquad (18)$$

Using equalities (12), (13), and (16)–(18) and the theorem on integration of the original function [2], we find the transforms:

$$L[Q_p(\mathrm{Fo})] = Q_0\Lambda^{-1}L[\mathrm{Fo}]\,,$$

$$L[Q_{h1}(\mathrm{Fo})] = \frac{2Q_0\exp(-\sqrt{s}) - Q_s[1+\exp(-2\sqrt{s})]}{s^2\left\{(1+\varepsilon_*\sqrt{s}-K_\varepsilon)\exp(-2\sqrt{s}) + (1+\varepsilon_*\sqrt{s}+K_\varepsilon)\right\}}\,; \qquad (19)$$

$$L[Q_{h2}(\mathrm{Fo})] = \frac{Q_0-Q_s}{s^2\left(K_\varepsilon\mu^{-1}s+(K_\varepsilon+\varepsilon_*)\sqrt{s}+1\right)}\,.$$

Thus, taking account of (14) and (19) and of the theorem on integration of the transform, we can state that

$$L[W_1(\mathrm{Fo},\gamma) - W_2(\mathrm{Fo},\gamma,\mu)] = L\left[\frac{Q_{h1}(\mathrm{Fo}) - Q_{h2}(\mathrm{Fo})}{Q_0\mathrm{Fo}\Lambda^{-1}}\right]$$

$$= \frac{\Lambda}{Q_0}\int_s^\infty\left\{\frac{2Q_0\exp(-\sqrt{p}) - Q_s[1+\exp(-2\sqrt{p})]}{\left\{(1+\varepsilon_*\sqrt{p}-K_\varepsilon)\exp(-2\sqrt{p}) + (1+\varepsilon_*\sqrt{p}+K_\varepsilon)\right\}} - \frac{Q_0-Q_s}{(K_\varepsilon\mu^{-1}p+(K_\varepsilon+\varepsilon_*)\sqrt{p}+1)}\right\}\frac{dp}{p^2}\,. \qquad (20)$$

In investigations with the use of integral transformations, one usually does not assume that one functional or another is the original function of the integral transform used [1–3, 12]. In this case, too, we have implicitly assumed that all the functionals in question are the original functions of the integral Laplace transform of a time variable Fo. Therefore, it is necessary to determine the condition of convergence of the improper integral on the right-hand side of the last equality (20), since only in this case will the difference of the integral values of the heat loss in the thermal protective coating of the system in question, which have been determined with the initial model (1)–(7) and its simplified analog (11), be the original function of the integral Laplace transform.

The improper integral on the right-hand side of the last equality (20) has a singularity at $s = 0 \in \mathbb{C}$ and allows, in the vicinity of this point, the following representation:

$$\int_s^{\infty} \left\{ \frac{\psi(p)}{\varphi(p)} - \frac{Q_0 - Q_s}{\omega(p)} \right\} \frac{dp}{p^2} = \int_s^{\infty} \frac{\psi(p)\,\omega(p) - (Q_0 - Q_s)\,\varphi(p)}{\varphi(p)\,\omega(p)\,p^2}\, dp ,$$

$$\psi(p) = 2Q_0 \exp(-\sqrt{p}) - Q_s\,[1 + \exp(-2\sqrt{p})]$$

$$= 2\,(Q_0 - Q_s) + \sum_{k=1}^{\infty} (-1)^k \frac{p^{k/2}}{k!}\,(2Q_0 - 2^k Q_s) ,$$

$$\varphi(p) = (1 + \varepsilon_* \sqrt{p} - \mathrm{K}_\varepsilon) \exp(-2\sqrt{p}) + (1 + \varepsilon_* \sqrt{p} + \mathrm{K}_\varepsilon)$$

$$= 2 + 2\,(1 + \varepsilon_* \sqrt{p} + \mathrm{K}_\varepsilon)\sqrt{p} + \sum_{k=2}^{\infty} (-1)^k p^{k/2} \frac{2^k}{k!} \left(1 - \mathrm{K}_\varepsilon - \frac{k}{2}\,\varepsilon_*\right),$$

$$\omega(p) = \mathrm{K}_\varepsilon \mu^{-1} p + (\varepsilon_* + \mathrm{K}_\varepsilon)\sqrt{p} + 1 .$$

Thus, in the vicinity of the singular point $s = 0 \in \mathbb{C}$, we have

$$\psi(p)\,\omega(p) - (Q_0 - Q_s)\,\varphi(p) \sim o\,(p\sqrt{p}) ,$$

and the improper integral in question converges only for

$$\mu = 2\left(1 - \frac{Q_s}{Q_0}\right)\mathrm{K}_\varepsilon , \qquad (21)$$

since otherwise, in the vicinity of the singular point $s = 0 \in \mathbb{C}$, we obtain

$$\psi(p)\,\omega(p) - (Q_0 - Q_s)\,\varphi(p) \sim o\,(p),$$

and the integrand is equivalent to the function p^{-1}.

Temperature Field. The temperature field of the isotropic half-space in the studied system isotropic half-space–thermoactive thermally thin spacer–thermal protective coating is determined by the functional $\theta_1(y, \mathrm{Fo})$ for $y \geq \varepsilon$, if the "accurate" model (1)–(7) is used, or by the functional $\theta_2(y, \mathrm{Fo})$ for $y \geq \varepsilon$, if the refined "lumped-capacity" model (11) is used. These functionals are the original functions of the integral Laplace transform (16) for the images $U_1(y, s)$ and $U_2(y, s)$ prescribed by equalities (17) and (18) respectively when $y \geq \varepsilon$.

The image $U_1(y, s)\big|_{y\geq\varepsilon}$ prescribed by equality (17) has a unique singular point $s = 0 \in \mathbb{C}$, which is a branch point. Therefore, for passage to the original function $\theta_1(y, \mathrm{Fo})\big|_{y\geq\varepsilon}$ it is sufficient to use the procedure of computation of the Mellin integral from the standard closed loop enclosing the branch point $s = 0$ [2]:

$$\theta_1 (y, \mathrm{Fo})\big|_{y\ge\varepsilon} = \frac{2}{\pi}\int_0^{+\infty} \frac{[Q_0 - Q_s \cos(\rho)]}{\cos^2(\rho) + [\rho\varepsilon_* \cos(\rho) + \mathrm{K}_\varepsilon \sin(\rho)]^2} \Big\{\cos[(y-\varepsilon)\rho]\cos(\rho)$$

$$- [y\varepsilon_* \cos(\rho) + \mathrm{K}_\varepsilon \sin(\rho)] \sin[(y-\varepsilon)\rho]\Big\} \frac{1-\exp(-\rho^2 \mathrm{Fo})}{\rho^2} d\rho , \quad \mathrm{Fo} \ge 0 . \tag{22}$$

Also, the original function $\theta_1(y, \mathrm{Fo})$ for $y \ge \varepsilon$ can be represented in the form of the sum of a uniformly convergent functional series. Indeed, according to (17), we obtain

$$U_1 (y, s)\big|_{y\ge\varepsilon} = \sum_{k=0}^{\infty} U_{1k} (y, s) ;$$

$$U_{1k} (y, s) = \frac{(-1)^k}{\varepsilon_*} \frac{[\sqrt{s} + (1-\mathrm{K}_\varepsilon)\,\varepsilon_*^{-1}]^k}{s\sqrt{s}\,[\sqrt{s} + (1+\mathrm{K}_\varepsilon)\,\varepsilon_*^{-1}]^{k+1}}$$

$$\times \Big\{2Q_0 \exp[-(y-\varepsilon+2k+1)\sqrt{s}] - Q_s [1+\exp(-2\sqrt{s})] \exp[-(y-\varepsilon+2k)\sqrt{s}]\Big\} , \quad y \ge \varepsilon , \quad k \ge 0 . \tag{23}$$

Thus, if $\theta_{1k}(y, \mathrm{Fo})$ is the original function of the transform $U_{1k}(y, s)$ for $y \ge \varepsilon$, by virtue of the uniform convergence of the expansion used and the linearity of the operator of reversal of the integral Laplace transform [2], we have

$$\theta_1 (y, \mathrm{Fo}) = \sum_{k=0}^{\infty} \theta_{1k} (y, \mathrm{Fo}) , \quad y \ge \varepsilon , \quad \mathrm{Fo} \ge 0 . \tag{24}$$

The main difficulty of realization of the approach described is that the original function of the transform $[s\sqrt{s}(\sqrt{s}+b)^n]^{-1} \exp(-k\sqrt{s})$, $n \in \{0, 1, 2, ...\}$ of interest is known just for $n = 0$ and **n** = 0 [2]. This original function can be found for any $n \ge 2$ with the use of the existing theorems of operational calculus [2]. Thus, e.g., when $n = 2$, orienting ourselves to the use of the particular case of the Efros theorem [2]

$$F(s) = L[f(t)] \Rightarrow L^{-1}\left[\frac{F(\sqrt{s})}{\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}} \int_0^{\infty} \exp\left(-\frac{\rho^2}{4t}\right) f(\rho)\, d\rho , \tag{25}$$

we have

$$\frac{F(\sqrt{s})}{\sqrt{s}} = \frac{\exp(-k\sqrt{s})}{\sqrt{s}\,(\sqrt{s})^2\,(\sqrt{s}+b)^2} \Rightarrow F(s) = \frac{\exp(-ks)}{s^2 (s+b)^2} .$$

In the case in question $s_{01} = 0$ and $s_{02} = -b$ are the zeros of second order for the denominator of the transform $F(s)$ and, according to [2],

$$L^{-1}\left[\frac{1}{s^2 (s+b)^2}\right] = \lim_{s\to 0} \frac{d}{ds} \left\{s^2 \frac{\exp(st)}{s^2 (s+b)^2}\right\} + \lim_{s\to -b} \frac{d}{ds} \left\{(s+b)^2 \frac{\exp(st)}{s^2 (s+b)^2}\right\} = b^{-3} [bt - 2 + (bt+2)\exp(st)] .$$

Using the retardation theorem [2], we find

$$f(t) = L^{-1}\left[\frac{\exp(-ks)}{s^2 (s+b)^2}\right] = b^{-3} [b(t-k) - 2 + [b(t-k)+2] \exp(-b(t-k))]\, \eta(t-k) .$$

To obtain the final result it is sufficient to substitute the original function found into the right-hand side of equality (25) and compute the corresponding integrals:

$$L^{-1}\left[\frac{\exp(-k\sqrt{s})}{s\sqrt{s}\,(\sqrt{s}+b)^2}\right]=\frac{4}{b^2}\sqrt{\frac{t}{\pi}}\exp\left(-\frac{k^2}{4t}\right)-\frac{bk+2}{b^3}\operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right)-\frac{2tb^2+bk-2}{b^3}$$

$$\times\exp(b^2t+bk)\operatorname{erfc}\left(\frac{k}{2\sqrt{t}}+b\sqrt{t}\right).$$

In passing from the transform $U_2(y, s)\big|_{y\geq\varepsilon}$ prescribed by equality (18) to the original function $\theta_2(y, s)\big|_{y\geq\varepsilon}$, we can have three different cases depending on the roots of the quadratic equation $z^2+[\mu(\mathrm{K}_\varepsilon+\varepsilon_*)/\mathrm{K}_\varepsilon]z+\mu/\mathrm{K}_\varepsilon = 0$ [2, 11]. In particular, for the temperature $\theta_2(\varepsilon+0, \mathrm{Fo})$ of the heat-insulated surface of the shielded half-space, we have the following representations:

$$\mu > 4\mathrm{K}_\varepsilon/(\mathrm{K}_\varepsilon+\varepsilon_*)^2 \Rightarrow \theta_2(\varepsilon+0, \mathrm{Fo})$$

$$=\frac{\mu(Q_0-Q_s)}{\mathrm{K}_\varepsilon(b_1-b_2)}\sum_{k=1}^{2}(-1)^k\left\{-\frac{1}{b_k^2}+2\sqrt{\frac{\mathrm{Fo}}{\pi}}\frac{1}{b_k}+\frac{1}{b_k^2}\exp(b_k^2\mathrm{Fo})\operatorname{erfc}(b_k\sqrt{\mathrm{Fo}})\right\} \qquad (26)$$

for $b_k = \dfrac{\mu(\mathrm{K}_\varepsilon+\varepsilon_*)}{2\mathrm{K}_\varepsilon}+(-1)^k\dfrac{\sqrt{\mu^2(\mathrm{K}_\varepsilon+\varepsilon_*)^2-4\mu\mathrm{K}_\varepsilon}}{2\mathrm{K}_\varepsilon}$, $k = 1, 2$;

$$\mu = 4\mathrm{K}_\varepsilon/(\mathrm{K}_\varepsilon+\varepsilon_*)^2 \Rightarrow \theta_2(\varepsilon+0, \mathrm{Fo})$$

$$=\frac{2\mu(Q_0-Q_s)}{\mathrm{K}_\varepsilon}\left(-\frac{1}{b^3}+2\sqrt{\frac{\mathrm{Fo}}{\pi}}\frac{1}{b^2}+\frac{1-b\mathrm{Fo}}{b^3}\exp(b^2\mathrm{Fo})\operatorname{erfc}(b\sqrt{\mathrm{Fo}})\right)$$

for $b = \dfrac{\mu(\mathrm{K}_\varepsilon+\varepsilon_*)}{2\mathrm{K}_\varepsilon}$;

$$\mu < 4\mathrm{K}_\varepsilon/(\mathrm{K}_\varepsilon+\varepsilon_*)^2 \Rightarrow \theta_2(\varepsilon+0, \mathrm{Fo}) = 2(Q_0-Q_s)\left\{\sqrt{\frac{\mathrm{Fo}}{\pi}}-\frac{\alpha}{\sigma}+\frac{1}{\beta\sigma}\exp[(\alpha^2-\beta^2)\mathrm{Fo}]\right.$$

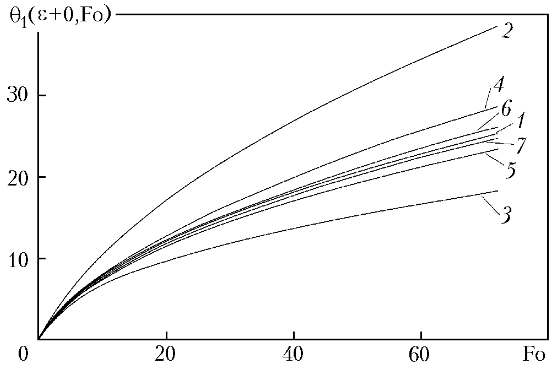
$$\times\left[\operatorname{erfc}(\alpha\sqrt{\mathrm{Fo}})\left(\alpha\beta\cos(2\alpha\beta\,\mathrm{Fo})-\frac{\alpha^2-\beta^2}{2}\sin(2\alpha\beta\,\mathrm{Fo})\right)+\frac{2}{\sqrt{\pi}}\exp(-\alpha^2\mathrm{Fo})\right.$$

$$\left.\left.\times\int_0^{\beta\sqrt{\mathrm{Fo}}}\exp(y^2)\left(\alpha\beta\sin[2\alpha\sqrt{\mathrm{Fo}}(\beta\sqrt{\mathrm{Fo}}-y)]+\frac{\alpha^2-\beta^2}{2}\cos[2\alpha\sqrt{\mathrm{Fo}}(\beta\sqrt{\mathrm{Fo}}-y)]\right)dy\right]\right\},$$

for $\alpha = \dfrac{\mu(\mathrm{K}_\varepsilon+\varepsilon_*)}{2\mathrm{K}_\varepsilon}$, $\beta = \dfrac{\sqrt{4\mu\mathrm{K}_\varepsilon-\mu^2(\mathrm{K}_\varepsilon+\varepsilon_*)^2}}{2\mathrm{K}_\varepsilon}$, and $\sigma = \dfrac{\mu}{\mathrm{K}_\varepsilon}$.

Results and Discussion. According to equality (21), the value of the determining parameter μ of the simplified analog of the initial mathematical model (1)–(7) is in direct proportion to the similarity simplex K_ε which is a measure of the relation of the thermal activities of the thermal protective coating and the isotropic half-space.

Theoretically we can have the following three situations: (1) $\mu < 0$, the specific power of heat absorption in the thermoactive spacer exceeds the density of the external heat flux, (2) $\mu = 0$, the thermoactive spacer totally com-



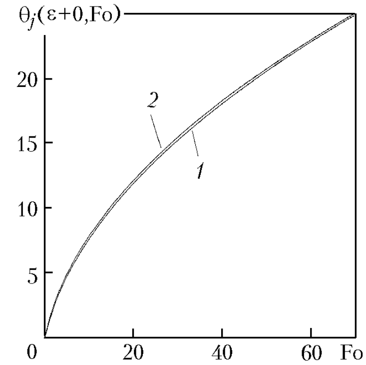


Fig. 2. Time dependences of the dimensionless temperature of the heat-insulated surface of a shielded half-space ($\varepsilon_* = 1$, $K_\varepsilon = 0.23$, $Q_0 = 2$, and $Q_s = -1$) determined with the accurate model according to equalities (22) (curve 1) and (23) and (24) for $y = \varepsilon + 0$ (2) $k = 0$, 3) 1, 4) 2, 5) 3, 6) 4, and 7) 5.

Fig. 3. Time dependences of the dimensionless temperature $\theta_j(\varepsilon + 0, \text{Fo})$ of the heat-insulated surface of a shielded half-space ($\varepsilon_* = 1$, $K_\varepsilon = 0.23$, $Q_0 = 2$, and $Q_s = -1$) determined with the accurate model ($j = 1$) according to equality (22) for $y = \varepsilon + 0$ (curve 1) and its simplified analog ($j = 2$) with the use of equality (26) (curve 2).

pensates for the external heat flux, and, (3) $\mu > 0$, the thermoactive spacer only attenuates the action of the external heat flux.

The temperature field in the shielded half-space with a thermoactive spacer under external thermal action can be found with the integral-transformation method. In the case in question it is determined by equality (22). However, practical realization of the obtained representation of the initial problem involves both overcoming certain difficulties of the computational character and a considerable consumption of time.

The functional determining the studied temperature field can be represented as the sum of a uniformly convergent functional series. But the employment of this approach involves two negative points: (1) difficulties arising in parametric analysis of the process of formation of the temperature field in the system studied; (2) the necessity of finding the original functions for integral Laplace transforms absent from the handbooks of operational calculus. Furthermore, the rate of convergence of the functional series used leaves much to be desired (Fig. 2). At the same time, we can be assured that the exact solution of the problem in question lies between the found particular sums of this functional series of the orders k and $k + 1$.

The representation of (26) of the temperature field studied that has been obtained with the simplified analog (10) (refined "lumped-capacity" model) of the initial mathematical model (1)–(7) is very convenient for practical realization. An analysis of the results of computational experiments partially illustrated in Fig. 3 enables us to assume that we can use the obtained simplified analog of the initial mathematical model in practice, for which purpose we must only determine the domain of permissible values of the vector of its parameters.

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NOTATION

a, thermal diffusivity, m^2/sec; $\mathbb{C}$, set (field) of complex numbers; Fo, Fourier number; erfc($\cdot$), complementary Gauss error function; i, imaginary unit; $L[\cdot]$ and $L^{-1}[\cdot]$, operators of direct and inverse integral Laplace transforms; $L^2[\varepsilon, +\infty)$, linear space of functions integrable with square on the semiinfinite interval $[\varepsilon, +\infty)$; h, dimensionless thickness of the thermal protection coating; K_ε, similarity simplex as a measure of the relation of the thermal activities of the thermal protective coating and the half-space; l_2, thickness of the thermal protective coating, m; l_3, thickness of the thermoactive spacer, m; $o(p)$, infinitesimal of the order p; Q_0, dimensionless density of the external heat flux; Q_s, dimensionless specific power of heat absorption of the thermally thin spacer; q_0, density of the external heat flux, W/m^2; $s \in \mathbb{C}$, parameter of integral Laplace transformation; T, temperature, K; t, time, sec; y, dimensionless space coordinate; z, space coordinate, m; z_*, selected unit of scale, m; ε, dimensionless thickness of the thermoactive spacer; ε_*, dimensionless criterial parameter; $\eta(\cdot)$, Heaviside unit function; θ, dimensionless temperature; $\langle\theta\rangle$, mean-integral temperature over the thickness ε of the thermoactive spacer; $\bar{\theta}$, mean-integral temperature over the thickness h of the thermal protective coating; Λ, dimensionless parameter characterizing the relative thermal conductivity of the thermal protective coating; λ, thermal conductivity, W/(m·K); χ^2, dimensionless parameter characterizing the thermoinertial properties of the thermal protective coating relative to the half-space; μ, identified dimensionless parameter of the refined "lumped-capacity" model. Subscripts: s, spacer; p, put-in; h, shielded half-space; t, transposition.

REFERENCES

1. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* [Russian translation], Nauka, Moscow (1964).
2. A. V. Luikov, *Heat-Conduction Theory* [in Russian], Vysshaya Shkola, Moscow (1967).
3. É. M. Kartashov, *Analytical Methods in the Theory of Heat Conduction of Solids* [in Russian], Vysshaya Shkola, Moscow (2001).
4. É. M. Kartashov, Analytical methods of solution of boundary-value problems of nonstationary heat conduction in regions with moving boundaries, *Inzh.-Fiz. Zh.*, **74**, No. 2, 171–195 (2001).
5. V. S. Zarubin, *Temperature Fields in the Structure of Flying Vehicles* [in Russian], Mashinostroenie, Moscow (1978).
6. Yu. V. Polezhaev and F. B. Yurevich, *Heat Shielding* [in Russian], Énergiya, Moscow (1976).
7. V. S. Zarubin, *Calculation and Optimization of Thermal Insulation* [in Russian], Énergoatomizdat, Moscow (1991).
8. Yu. V. Polezhaev, S. V. Reznik, É. B. Vasilevskii, et al., *Materials and Coatings under Extreme Conditions. Look into the Future*, in 3 vols., Vol. 1, *Prediction and Analysis of Extremal Effects* [in Russian], Izd. MGTU im. N. É. Baumana, Moscow (2002).
9. E. K. Iordanishvili, *Thermoelectric Power Supplies* [in Russian], Sovetskoe Radio, Moscow (1968).
10. E. I. Kim, V. T. Omel'chenko, and S. N. Kharin, *Mathematical Models of Thermal Processes in Electric Contacts* [in Russian], Nauka, Alma-Ata (1977).
11. A. V. Attetkov, I. K. Volkov, and E. S. Tverskaya, A thermoactive gasket as a means of controlled influence on the temperature field of a structure, *Izv. Ross. Akad. Nauk, Énergetika*, No. 4, 131–141 (2002).
12. M. A. Pudovkin and I. K. Volkov, *Boundary-Value Problems of the Mathematical Theory of Heat Conduction as Applied to Calculations of Temperature Fields in Oil Beds on Flooding* [in Russian], Izd. Kazansk. Univ., Kazan' (1978).
13. A. V. Attetkov, P. A. Vlasov, and I. K. Volkov, Temperature field of a half-space with a thermally thin coating in pulse modes of heat exchange with the environment, *Inzh.-Fiz. Zh.*, **74**, No. 3, 81–86 (2001).
14. P. A. Vlasov, Mathematical simulation of a temperature field in a half-space with a heat-shielding coating, in: *The Physical Principles of Experimental and Mathematical Modeling of the Processes of Gas Dynamics and Heat/Mass Transfer in Power Plants*, *Proc. of the 13th School-Seminar of Young Scientists and Specialists under the Supervision of Academician A. I. Leontiev* [in Russian], Vol. 2, Izd. MÉI, Moscow (2001), pp. 166–169.
15. A. V. Attetkov, I. K. Volkov, and E. S. Tverskaya, Mathematical simulation of the process of heat transfer in a screened wall with axisymmetric thermal effect, *Izv. Ross. Akad. Nauk, Énergetika*, No. 5, 75–881 (2003).